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Relativistic Ornstein–Uhlenbeck Process

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We wish to shed some light on the problem of thermodynamic irreversibility in the relativistic framework. Therefore, we propose a relativistic stochastic process based on a generalization of the usual Ornstein–Uhlenbeck process: we introduce a relativistic version of the Langevin equation with a damping term which has the correct Galilean limit. We then deduce relativistic Kramers and Fokker–Planck equations and a fluctuation-dissipation theorem is derived from them. Finally, numerical simulations are used to check the equilibrium distribution in momentum space and to investigate diffusion in physical space.

KEY WORDS: Irreversibility; Einstein's relativity; Ornstein–Uhlenbeck process.

1. NOTATION

In this article, the velocity of light will be denoted by c and the signature of the space time metric will be chosen to be $(+, -, -, -)$. γ and m are respectively the Lorentz factor and the mass of the particle undergoing stochastic motion. T will designate the absolute temperature, and k will stand for the Boltzmann constant. The second order modified Hankel function[(1)] will be denoted $K_2(x)$. As usual, Greek indices will run from 0 to 3 and Latin ones from 1 to 3, except in Appendix A, where Latin indices run from 1 to 6.

2. INTRODUCTION

Modeling and understanding irreversible behavior has been one of the key jobs of Thermodynamics and Statistical Physics. The first approach

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to irreversibility has its roots in the Thermodynamical theories of homogeneous phases[2] and developed into Galilean irreversible hydrodynamics;[3] this set of theories has been more recently the object of renewed conceptual and practical interest[4] and have finally been given relativistic generalizations which do not violate causality.[5] However, these relativistic versions and their Galilean counterparts predict uncustomary shock waves structures which do not seem to have been observed experimentally.[6] We therefore believe that the foundations of irreversible relativistic hydrodynamics may still be considered a potentially interesting field of study.

Another, purely statistical way to comprehend irreversible behavior is best represented by Boltzmann's equation,[7] of which there also exists a relativistic version. However, to the best of our knowledge, this relativistic version cannot be obtained by truncating a relativistic equivalent of the BBGKY hierarchy; indeed, such an equivalent does not seem to have been proposed yet (at least if the interactions are conceived in the framework of a field theory). The third and last approach to Galilean irreversible phenomena has been the study of stochastic processes, originated by Einstein's work on Brownian motion.[8, 9] The present paper is devoted to the presentation of a possible relativistic generalization of the usual Galilean results on the Ornstein–Uhlenbeck process.

In Section 3, we introduce the theoretical fundamentals underlying our work; we first construct explicitly a relativistic covariant stochastic process, the Galilean limit of which is the usual Ornstein–Uhlenbeck process; we then proceed by writing down Kramers' and Fokker-Planck equations which are verified by this process and prove a corresponding fluctuation-dissipation theorem; Section 4 introduces relevant numerical simulations; in the conclusion, we finally review some of the remaining problems left open for further study. Appendix A presents a detailed derivation of Kramers' equation for the stochastic process we consider in this article and Appendix B contains some algebraic computations about the Jüttner distribution introduced in Section 3.2.

3. GENERAL THEORETICAL FRAMEWORK

3.1. Relativistic Langevin Equation

Einstein's original theory of Brownian motion[8, 9] cannot actually be extended to a formalism compatible with Special or General Relativity; the reason is that, if one follows Einstein's approach, the space and time-coordinates play fundamentally different roles and it would be completely impossible, for example, to consider a "motion" in which the particle has the possibility to jump forward or backward in space and in time. However, Ornstein's and Uhlenbeck's presentation of Brownian motion,[10–12] which derives it from the motion of a particle submitted to the action of both a stochastic and a deterministic (damping) force is suitable for relativistic generalization. In what follows, we will focus our discussion on the special relativistic case and only comment briefly on the general relativistic one.

The first ingredient one needs to write down a relativistic Langevin equation is an expression for the deterministic part f of the force which appears on the right-hand side of the equation. We postulate an ansatz for f, imposing that it be a 4-vector and that it has the correct Galilean limit to be found in the usual Langevin equation. Let u and U be respectively the 4-velocities of the moving particle and of the fluid surrounding it. We introduce a tensor λ, the properties of which will be discussed below, and propose the following expression for f:

$$f^\mu = -m\lambda^\mu_\nu(u^\nu - U^\nu) + m\lambda^\alpha_\beta u_\alpha(u^\beta - U^\beta)\, u^\mu \tag{1}$$

The first contribution to the right-hand side of (1) has a clear intuitive meaning and the second one ensures that f is orthogonal to u so that the condition $u^2 = 1$ is not violated by the motion. In their rest frame, the simplest fluids do not exhibit any preferred space direction; we therefore postulate that for such fluids λ is of the form

$$\lambda^\mu_\nu = \chi U^\mu U_\nu + \alpha(\delta^\mu_\nu - U^\mu U_\nu) \tag{2}$$

where χ and α are two scalars which may depend on the thermodynamical state of the fluid in which the diffusion occurs, as well as on any scalar quantity built out of u, for example the modulus of the 3-velocity in the rest frame of the fluid. As a matter of fact, assuming the metric to be Minkovskian, one finds immediately that, in the rest frame of the fluid where $U = (1, \mathbf{0})$,[4] equation (2) gives:

$$\lambda^\mu_\nu = \begin{pmatrix} \chi & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

[4] From hereon, we shall tacitly assume that the surrounding fluid is in an equilibrium state, so that its rest-frame is not only local but also global; note however that (1) and (2) remain valid even if this is not the case.

which does not particularizes any spatial direction. Unfortunately, the role of χ does not seem to have a straightforward general interpretation; moreover, one can check directly that (1) gives back the usual Galilean limit for the deterministic damping force i.e., a linear expression in the (3-)velocity of the particle, with a coefficient equal to the limit value of α for small velocities; besides χ makes no contribution to this regime if it remains a finite quantity. We will henceforth, for simplicity reasons only, restrict the developments presented in this article to the case where χ identically vanishes; a thorough study of the influence of this coefficient on the relativistic diffusion problem will be presented elsewhere. Since f is introduced as the damping contribution to the force which acts on the particle, one imposes that, under the action of f, the energy and the velocity of the particle in the rest frame of the fluid have to decrease in time (and tend respectively towards the rest mass energy and zero). This implies through a straightforward calculation that α has to be strictly positive, provided the metric in this frame is Minkovskian. The physically correct expression for α is given by the fluctuation-dissipation theorem derived in Section 3.2. Let us finally stress that (1) and (2) do not presuppose the space-time to be flat and are also appropriate for a general relativistic treatment.

Let us now introduce a "random force" $F = (F_0, \gamma \mathbf{F}/c^2)$, where the first component F_0 is determined by the orthogonality of F and u, without worrying about any precise mathematical definition of this concept at this stage. Adding the contribution of F to the one of the frictional force f and supposing the global rest frame of the fluid to be inertial, one obtains in this reference frame:

$$\frac{d\mathbf{p}}{dt} = -\alpha\gamma\mathbf{p} + \mathbf{F} \tag{3}$$

which is the special relativistic equivalent of Langevin equation. For simplicity reasons, it is rather usual, in the Galilean case, to give a mathematically well-defined meaning to the concept of random force by defining the Galilean random contribution to the force experienced by the particle in the rest frame of the fluid to be a so-called "Gaussian white noise" or, equivalently, the derivative of a Wiener process with vanishing mean value. Such a modeling of F in the relativistic regime does not seem physically absurd and presents exactly the same technical advantages as the one retained in the Galilean limit. We will thus define the random force F such that the corresponding 3-force $\mathbf{F}$ in the rest frame of the fluid is a centered Gaussian white noise verifying:

$$\langle \mathbf{F}(t) \rangle = \mathbf{0} \tag{4}$$

$$\langle F^i(t_1)\, F_j(t_2) \rangle = -2D\delta(t_2 - t_1)\, \delta^i_j, \qquad D > 0 \tag{5}$$

Of course, this does not ensure that, in another reference frame, the 3-force **F** will also have the same properties, except naturally in the Galilean limit. This is naturally no problem for the covariance of the model we present. The whole situation under study exhibits a preferred reference frame and so should any formalism which describes it. The situation is somewhat similar to the case of an electromagnetic field which degenerates to a pure electric one in a preferred reference frame but which appears as both electric and magnetic in any other reference frame. The model we propose is covariant, albeit not manifestly, because, once we know its properties in a reference frame, we are able to construct (and study) it in any other one. A simple example of a non manifestly covariant formalism is the Hamiltonian approach to field theory,[13, 14] which singles out the time-coordinate but which is nevertheless perfectly meaningful and consistent with Einstein's principle of relativity.

Some additional non-technical comments on f and the process as a whole may be useful before we embark on further analytical calculations. The point we would like to emphasize is that f does not represent the real frictional force which acts on a relativistic particle in motion in a dissipative (relativistic) fluid. The reason for this is that a particle moving with a high enough velocity for (special) relativistic effects to be of importance would generate a high Reynolds number flow in the surrounding fluid and a simple one-to-one link between the velocity of the particle and the frictional force experienced is certainly out of the question. What is then the status of the Ansatz (1)? It is a deterministic Lorentz-invariant term to be used in conjunction with the noise contribution F in order to generate a stochastic process which will (i) exhibit irreversible behavior, (ii) display a trend towards the correct relativistic equilibrium (expressed notably by a fluctuation-dissipation theorem), (iii) have the usual Ornstein–Uhlenbeck process as Galilean limit. In particular, (1) should not be given a direct physical meaning outside the construct of the stochastic process we are currently describing. This means that the stochastic process (3) is not to be considered as a theory for the diffusion of a real relativistic colloidal particle immersed in a real (relativistic) fluid. It is just a toy-model of relativistic irreversible behavior. What can it be useful to? It can help test the general structure of a macroscopic relativistic fluid theory. Indeed, as in the Galilean case, one can put in formal correspondence various mean values generated by our stochastic process with fields which appear naturally in continuous descriptions of fluids (for example, the particle four-current and the stress-energy tensor). A reasonable requirement on any relativistic fluid dynamics seems to be that its mathematical structure can at least accommodate the dynamics of these mean values, as deduced from Kramers' equation (see Section 3.2).[5] Preliminary calculations show that this test is very stringent; this provides a justification for the construction of the model we present in this article.

It may not be uninteresting to note at this point that the interpretation of the deterministic part of the force which acts on the particle as a real frictional force is not free of problems, even in Galilean Physics: it can be proven that, in the Galilean regime,[16] if the random force which acts on the particle is a Gaussian white noise, the frictional force has to be linear in the velocity in order for the equilibrium distribution function in momentum space to be Maxwellian. This is an exact result, totally independent from the Stokes calculation of the drag force experienced by a sphere in slow motion in a dissipative fluid. As a matter of fact, if one includes in the Galilean Langevin equation the corrections to Stokes' law due to Oseen,[17] Kaplun and Lagerstrom,[18] Proudman and Pearson,[19] the corresponding equilibrium distribution function is no longer Maxwellian; this clearly shows that, even in Galilean Physics, the deterministic part of the force used in constructing a Langevin-type stochastic process is not the real frictional force experienced by a moving object in the fluid under consideration. Naturally, as we already mentioned before, the connection between the deterministic part of the force in (3) and a real frictional force is even looser in the relativistic realm since the very notion of a velocity-dependent frictional force is highly suspect in the (special) relativistic regime. This however does not invalidate the stochastic process (3) as a toy-model of relativistic irreversible behavior.

We will devote the rest of this work to a more precise study of the special relativistic process (3) in its preferred reference frame, where (3) holds and $\mathbf{F}$ is a particularly simple mathematical object. The evolution of the same process in other reference frames will be addressed in future publications.

3.2. Relativistic Kramers' Equation

Having chosen our reference frame, let us now introduce (in this frame) a distribution function $\Pi(t, \mathbf{x}, \mathbf{p})$ in phase space, associated to the usual measure $d^3x\, d^3p$. From the stochastic differential equation (3) a "diffusion equation" for Π can be deduced[11,20] (see Appendix A for the derivation):

$$\frac{\partial \Pi}{\partial t} + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{p}}{\gamma m} \Pi\right) + \nabla_{\mathbf{p}} \cdot (-\alpha\gamma\mathbf{p}\Pi) = D\Delta_{\mathbf{p}}\Pi \tag{6}$$

[5] The corresponding Galilean test is passed for example by the usual diffusion theory: Fick's law can be derived from a systematic expansion of the Galilean Kramer's equation.[3,15]

the Galilean equivalent of which is known as Kramers' equation. Naturally, (6) can also be integrated over the physical three-dimensional space to deliver a Fokker-Planck equation for the marginal $\tilde{\Pi}(t, \mathbf{p})$ of Π:

$$\frac{\partial \tilde{\Pi}}{\partial t} + \nabla_{\mathbf{p}} \cdot (-\alpha \gamma \mathbf{p} \tilde{\Pi}) = D \Delta_{\mathbf{p}} \tilde{\Pi} \tag{7}$$

The general solution to either (6) or (7) cannot be found analytically but the study of Kramers' equation can nevertheless deliver interesting analytical results on the diffusion process in phase space. As an example, let us search for a stationary, spatially uniform and isotropic solution Π_{eq} to (6). It follows directly from (6) that there exists a field $\mathbf{A}(\mathbf{p})$ linked to Π_{eq} by the relation:

$$\nabla_{\mathbf{p}} \times \mathbf{A} = -\alpha \gamma \mathbf{p} \Pi_{eq} - D \nabla_{\mathbf{p}} \Pi_{eq} \tag{8}$$

If one assumes temporarily that $\mathbf{A}$ identically vanishes, then (8) becomes:

$$D \nabla_{\mathbf{p}} \Pi_{eq} = -\alpha \gamma \mathbf{p} \Pi_{eq} \tag{9}$$

This equation can be considered from two complementary points of view: One can either solve it in Π_{eq} for a given α or find the correct α corresponding to an *a priori* given Π_{eq}. As an example of the first possibility, the simplest choice is to take α constant: $\alpha = \alpha_0$; this gives, according to (9), the following somewhat unphysical form for Π_{eq}:

$$\Pi_{eq}(\mathbf{p}) = \Pi_0 \exp\left(-\frac{\alpha_0 m^2 c^2}{3D} \gamma^3\right) \tag{10}$$

where Π_0 is fixed by the normalization condition:

$$\int_{\mathbb{R}^3} \Pi_{eq}(\mathbf{p})\, d^3 p = 1 \tag{11}$$

Conversely, if one chooses for Π_{eq} the usual relativistic Maxwell-Boltzmann distribution[6, 21] (see Appendix B for the normalization factor):

$$\Pi_{eq}(\mathbf{p}) = \frac{1}{4\pi (mc)^3} \frac{mc^2/kT}{K_2(mc^2/kT)} \exp\left(-\frac{mc^2}{kT} \gamma\right) \tag{12}$$

then (9) delivers α:

$$\alpha = \frac{D}{mkT} \frac{1}{\gamma^2} = \frac{\alpha_1}{\gamma^2} \tag{13}$$

This result constitutes the (special) relativistic equivalent of the usual Galilean fluctuation-dissipation theorem.[11, 12]

If we now relax the hypothesis that **A** vanishes, assuming only that the mean value of $\mathbf{p}^2$ exists, then it can be shown that the only equilibrium distribution functions compatible with the choices $\alpha = \alpha_0$ and $\alpha = \alpha_1/\gamma^2$ are respectively (10) and (12).

A more thorough understanding of relativistic effects can surely be gained through further analytical studies of Kramers' equation but this is certainly an even harder task than the equivalent Galilean problem. We will therefore leave a more complete analytical discussion to future publications and go on to presenting some numerical simulations.

4. NUMERICAL SIMULATIONS

The purpose of this early numerical study is (i) to evaluate the time needed for the probability distribution Π to reach its equilibrium state, (ii) to check that this computed equilibrium state meets the prediction of (9), (iii) to find the time-evolution of the mean square end-to-end displacement of the "Brownian" particle in the rest frame of the fluid.

To achieve these goals, we have integrated numerically a dimensionless version of the three-dimensional relativistic Langevin equation (3) with a γ-dependent friction coefficient of the form $\alpha = \alpha_1/\gamma^2$ with constant α_1. We have thus introduced the following dimensionless functions and variables, that we denote from now on by underlined letters:

$$\underline{t} = \alpha_1 t, \qquad \underline{\mathbf{x}} = \frac{\alpha_1}{c} \mathbf{x}, \qquad \underline{\mathbf{p}} = \frac{\mathbf{p}}{mc}, \qquad \underline{\mathbf{F}} = \frac{\mathbf{F}}{\sqrt{D\alpha_1}}$$

This comes back to choose the "diffusive time" α_1^{-1}, the light velocity c and the particle mass m as the natural units of time, velocity and mass respectively. Note that the dimensionless random force $\underline{\mathbf{F}}$ defined above is a centered Gaussian white noise verifying:

$$\langle \underline{\mathbf{F}}(\underline{t}) \rangle = \mathbf{0} \tag{14}$$

$$\langle \underline{F}^i(\underline{t}_1)\, \underline{F}_j(\underline{t}_2) \rangle = -2\delta(\underline{t}_2 - \underline{t}_1)\, \delta^i_j \tag{15}$$

In terms of these variables, the dimensionless Langevin equation reads:

$$\frac{d\underline{\mathbf{p}}}{d\underline{t}} = -\frac{\underline{\mathbf{p}}}{\sqrt{1+\underline{\mathbf{p}}^2}} + Q\underline{\mathbf{F}} \tag{16}$$

where Q is a dimensionless control parameter which governs the balance between the random force and the frictional force. Its expression in terms of the physical parameters α_1 and D of the model is:

$$Q = \sqrt{\frac{D}{\alpha_1 m^2 c^2}} \tag{17}$$

Using the fluctuation-dissipation theorem (13), which defines the notion of temperature in our model, we can relate directly the parameter Q to the temperature as follows:

$$Q = \sqrt{\frac{kT}{mc^2}} \tag{18}$$

The standard physical intuition for the Galilean case makes us expect that the higher Q is, the higher is the root mean square momentum of the particle. This is also the case in the relativistic regime, as proven in Appendix B. Thus, the Galilean regime (small values of $\underline{\mathbf{p}}$ r.m.s) corresponds to the limit of vanishing values of Q. More precisely, high values of $|\underline{\mathbf{p}}|$ are not forbidden even for low values of Q, but they become less and less probable when Q decreases, at least when equilibrium is reached.

The integration scheme is the well-known second order[6] Runge–Kutta integrator.[22]

For numerical convergence reasons (the numerical results have to converge when the time step tends to zero), the Gaussian random force has been implemented as a "quasi-white noise" with a finite correlation time $\underline{t}_c$ much smaller than the diffusive relaxation time $\underline{t}_d \equiv 1$. The time step $\underline{t}_s$ is itself chosen to be very small compared to the correlation time $\underline{t}_c$, so that the random force "seems" continuous at the scale of a time step.

The Gaussian "quasi-white" random function $\underline{\mathbf{F}}$ is constructed as follows: the minimal standard random numbers generator of Park and Miller with Bays–Durham shuffling[22] is used three times in sequence to obtain three statistically independent uniformly distributed pseudo-random

[6] Some validation tests have also been performed with a first order Euler scheme and with a fourth order Runge–Kutta integrator, without any substantial change in the results.

variables. The Box–Muller algorithm[22] is then applied to transform these three uniform variables into three Gaussian variables, one for each component of the random 3-force $\underline{\mathbf{F}}$. These three random sequences are finally smoothed by convolution with a Gaussian function to introduce a finite correlation time.

The statistical quantities are computed by ensemble-averaging over several repeated simulations with the same initial condition, but with different random generator seeds. Furthermore, for an accurate computation of the equilibrium distribution function, we introduce an additional time-averaging over a carefully chosen domain in order to be sure that transient regimes have been fully damped.

In the simulation we now report on, the control parameter Q is 2.0. The time step and the correlation time are respectively $\underline{t}_s = 0.01$ and $\underline{t}_c = 0.1$. The Brownian particle is initially set at rest at the origin of coordinates and its evolution is computed up to $\underline{t} = 100$. The ensemble-averaging is done over 10000 samples.

The value $Q = 2.0$ of the control parameter is high enough to guarantee that the motion of the Brownian particle is not Galilean. Indeed, the probability distribution for $|\underline{\mathbf{p}}|$ has its maximum for $|\underline{\mathbf{p}}| \simeq 8.0$, which is clearly in the relativistic domain.

The simulation has revealed the following features:

- The probability distribution for $|\underline{\mathbf{p}}|$ reaches its equilibrium value before $\underline{t} = 10$.
- As shown by Fig. 1, the simulated equilibrium histogram corresponds, with a good precision, to the relativistic Maxwell–Boltzmann distribution (12) predicted by (9). The relative r.m.s. discrepancy between the computed histogram and the theoretical curve falls below 1%.
- The coefficient mc^2/kT of the best-fitting Maxwell–Boltzmann distribution is obtained by a least squares method and is found to be:

$$\frac{mc^2}{kT} = 0.259$$

which is close to the value 0.25 predicted by the relativistic fluctuation-dissipation theorem (13) for $Q = 2.0$. Note that a rapid but less accurate estimate of the coefficient mc^2/kT can also be obtained by computing the mean value of $(2\,\underline{\mathbf{p}}^2 + 1)/\gamma\underline{\mathbf{p}}^2$ with respect to the simulated equilibrium distribution. This leads to $mc^2/kT = 0.263$.

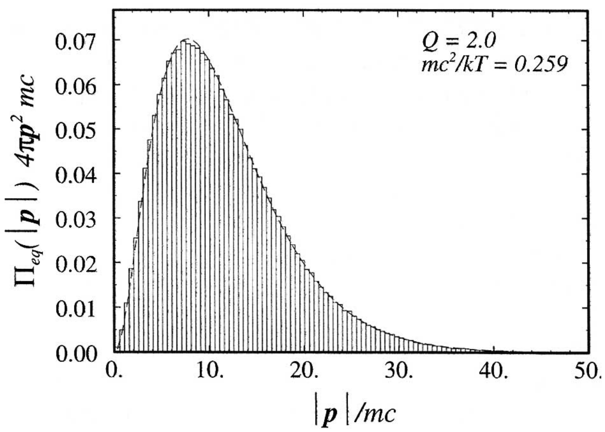

Fig. 1. Equilibrium distribution of $|\mathbf{p}|$, for $Q = 2.0$. The dashed curve corresponds to the best fitting relativistic Maxwell–Boltzmann distribution.

- After a transient regime ranging up to $\underline{t}_0 \simeq 40$, the time-evolution of the mean square dimensionless end-to-end displacement $\underline{\sigma}^2$ of the Brownian particle is very close to a linear growth (see Fig. 2). The slope of this linear time-evolution is found to be:

$$\frac{\underline{\sigma}^2(\underline{t}) - \underline{\sigma}^2(\underline{t}_0)}{\underline{t} - \underline{t}_0} = 17.5$$

in terms of the dimensionless space and time variables.

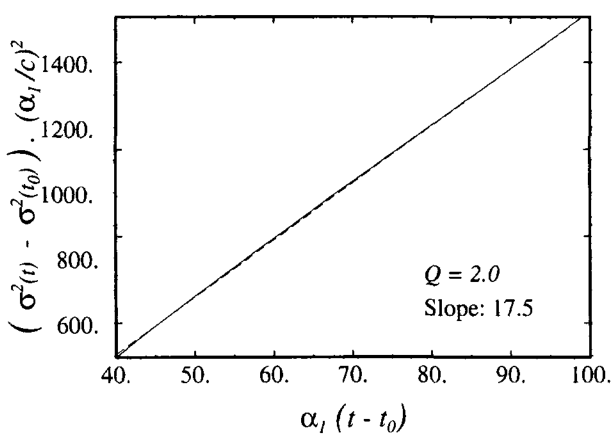

Fig. 2. Mean square displacement versus time, for $Q = 2.0$. The dashed curve corresponds to the best fitting straight line.

The simulations results discussed above are in good agreement with the theoretical predictions of Section 3, and also suggest that the classical linear growth of the mean square displacement with time still holds in the relativistic case.

Further simulations with different values of the control parameter Q and covering larger time ranges are in progress, to study the dependence on Q of the transient regime.

5. CONCLUSION

In this paper, we have constructed a toy-model which extends to the relativistic framework the formal structure of Ornstein's and Uhlenbeck's presentation of a Brownian motion. We have introduced a general relativistic generalization of Langevin's equation which has the correct Galilean limit. We have then restricted our scope to the special relativistic case, choosing as a reference frame the (global) rest frame of the surrounding fluid. We have introduced in this frame a Kramers' and a Fokker–Planck equation with which the stochastic process can best be studied. In particular, this formalism enabled us to obtain a (special) relativistic fluctuation-dissipation theorem. Other relativistic fluctuation-dissipation theorems have been proposed recently.[23] These results are not to be compared with ours. Indeed they have been derived from a (relativistic) stochastic Boltzmann equation and deliver the correlation functions of the stochastic parts of various macroscopic fields. On the contrary, our approach introduces stochasticity only at the level of the force experienced by single particles, and the mean values corresponding to the various macroscopic fields studied in ref. 23 do not have a stochastic part. In other words, Reference 23 deals with the fluctuations of the macroscopic fields around their mean value, whereas we have considered fluctuations of the force experienced by the particles in stochastic motion.

We have presented some relevant numerical simulations of the equations obtained in the first part of the paper; the numerical results seem to corroborate the fact that, in the rest frame of the fluid, the mean-square displacement of an assembly of diffusing particles originally concentrated in one single point in three-dimensional space varies linearly with time in the relativistic regime too, provided naturally that one waits long enough.

There are clearly a number of points which ask for clarification and which we are currently studying; the first set of interrogations is concerned with the possible extension of the results obtained in this article if the reference frame is inertial but does not coincide with the (global) rest-frame of the fluid and, more generally, if the chosen reference frame is not inertial; for example, what are then the properties of the random three-force and what do Kramers' and Fokker–Planck equations look like in a general frame? A second question is just how much analytical information about the motion in physical space can be extracted from Kramers' equation. In particular, we are currently studying the relativistic diffusion in physical space by using the formalism presented in this work. A related line of inquiry is naturally to investigate which conclusions about momentum and energy transfers can be derived from our stochastic process. This will permit a systematic comparison of the "predictions" of our toy-model with the results derived from various relativistic theories of diffusion (for example, the most recent ones which have been developed in the general framework of Extended Thermodynamics[5]).

APPENDIX A. DERIVATION OF KRAMERS' EQUATION

This appendix presents a derivation of Kramers' equation, which is essential to the developments in Section 3.2. What follows is inspired from ref. 20.

We start from the stochastic system:

$$\begin{cases} d\mathbf{x} = \dfrac{\mathbf{p}}{m\gamma}\, dt \\ d\mathbf{p} = -\alpha(\mathbf{p})\, \gamma\mathbf{p}\, dt + \sqrt{2D}\, d\mathbf{w} \end{cases} \tag{A.1}$$

In (A.1), α is a continuous function of $\mathbf{p}$, γ stands for the Lorentz factor $\sqrt{\mathbf{p}^2/m^2c^2+1}$, and $\mathbf{w}(t)$ designates the three-dimensional Wiener process:

$$\mathbf{w}(t) = (w_1(t), w_2(t), w_3(t)) \tag{A.2}$$

where $w_i(t)$, $i = 1,..., 3$ are three independent one-dimensional Wiener processes which verify, by definition, the following properties:

- (i) $w_i(0) = 0$
- (ii) For any s and t such that $0 \leqslant s \leqslant t$, the random variable $w_i(t) - w_i(s)$ has the Gaussian density:

$$g(t-s, \omega) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{\omega^2}{2(t-s)}\right) \tag{A.3}$$

The stochastic system (A.1) fixes the time evolution of both the position $\mathbf{x}$ and momentum $\mathbf{p}$ of the particle.

To ease further demonstrations, we need to gather the two three-dimensional equations in (A.1) into one single six-dimensional equation.

We thus introduce the condensed notation $Z=(\mathbf{x}, \mathbf{p})$ to gather the position and momentum variables, and define the following quantities:

$$\Phi(Z)=\left(\frac{\mathbf{p}}{m\sqrt{\mathbf{p}^2/m^2c^2+1}}, -\alpha(\mathbf{p})\,\mathbf{p}\sqrt{\frac{\mathbf{p}^2}{m^2c^2}+1}\right)$$

$$W(t)=(\mathbf{0}, \mathbf{w}(t))$$

$$\mathscr{D}=\operatorname{diag}(0, 0, 0, \sqrt{2D}, \sqrt{2D}, \sqrt{2D})$$

The stochastic system then takes the form:

$$dZ=\Phi(Z)\,dt+\mathscr{D}\,dW \tag{A.4}$$

We introduce now $G(t, Z')=G(t, x'_1, x'_2, x'_3, p'_1, p'_2, p'_3)$, the density of the stochastic process $W(t)$. Since the three components of $\mathbf{w}(t)$ are statistically independent, the density $G(t, Z')$ is

$$G(t, Z')=\delta(x'_1)\,\delta(x'_2)\,\delta(x'_3)\;g(t, p'_1)\;g(t, p'_2)\;g(t, p'_3) \tag{A.5}$$

where $\delta(x'_i)$ designates the one-dimensional Dirac distribution of the variable x'_i. The density G has the following important properties, which will be used below:

$$\int_{\mathbb{R}^3} d^3x' \int_{\mathbb{R}^3} G(t, \mathbf{p}')\,d^3p'=1 \tag{A.6}$$

$$\int_{\mathbb{R}^3} d^3x' \int_{\mathbb{R}^3} p'_i G(t, \mathbf{p}')\,d^3p'=0, \qquad i=1,..., 3 \tag{A.7}$$

$$\int_{\mathbb{R}^3} d^3x' \int_{\mathbb{R}^3} p'_i p'_j G(t, \mathbf{p}')\,d^3p'=\delta_{ij}t, \qquad i, j=1,..., 3 \tag{A.8}$$

Equation (A.1) at time t_0 is the continuous limit of:

$$Z(t_0+\Delta t)=Z(t_0)+\Phi(Z(t_0))\,\Delta t+\mathscr{D}\,\Delta W(t_0) \tag{A.9}$$

where Δt is a small time steep, and $\Delta W(t_0)=W(t_0+\Delta t)-W(t_0)$. Note that the deterministic "force" $\Phi(Z)$ is a continuous function; it is therefore bounded on any compact set.

We now derive the forward evolution equation for the phase space distribution function $\Pi(t, \mathbf{x}, \mathbf{p})$. We suppose that Π is sufficiently regular for its first derivatives with respect to time and position and its second derivative with respect to momentum to exist.

Let h be a $\mathscr{C}^\infty$ real-valued test function with compact support included in $\mathbb{R}^6$. The idea behind the proof is to evaluate the same quantity in two different ways. On one hand, the mean value $\langle h\rangle(t_0+\Delta t)$ of the function h can be computed with respect to the measure defined by Π at $t_0+\Delta t$. On the other hand, one can also evaluate the expectation value $E(h(Z(t_0+\Delta t)))$ of the random variable $h(Z(t_0+\Delta t))$ at the same instant $t_0+\Delta t$ in terms of the distribution Π at t_0. Letting Δt tend to zero will yield the desired evolution equation for Π.

By definition, one can write:

$$\langle h\rangle(t_0+\Delta t)=\int_{\mathbb{R}^6} h(Z)\,\Pi(t_0+\Delta t, Z)\,dZ \tag{A.10}$$

On the other hand, $Z(t_0+\Delta t)$ depends on the two random variables $Z(t_0)$ and $\Delta W(t_0)$. These two variables are independent. Therefore the density of the pair $(Z(t_0), \Delta W(t_0))$ is the product $\Pi(t_0, Z)\,G(\Delta t, Z')$. One can therefore write:

$$E(h(Z(t_0+\Delta t)))=\int_{\mathbb{R}^6}\int_{\mathbb{R}^6} h(Z+\Phi(Z)\,\Delta t+\mathscr{D}Z')\,\Pi(t_0, Z)\,G(\Delta t, Z')\,dZ\,dZ' \tag{A.11}$$

The characteristic extension of G in Z'-space is of order $\sqrt{\Delta t}$. This suggests to use in the r.h.s of (A.11) the second order Taylor expansion of h:

$$h(Z+\varepsilon)=h(Z)+\varepsilon_i\left.\frac{\partial h}{\partial Z_i}\right|_Z+\frac{1}{2}\varepsilon_i\varepsilon_j\left.\frac{\partial^2 h}{\partial Z_i\,\partial Z_j}\right|_Z + \frac{1}{3!}\varepsilon_i\varepsilon_j\varepsilon_k\left.\frac{\partial^3 h}{\partial Z_i\,\partial Z_j\,\partial Z_k}\right|_{Z+a\varepsilon} \tag{A.12}$$

where the real number a lies in $[0, 1]$, and where the implicit summation over repeated indices (ranging from 1 to 6) is used. One gets consequently:

$$\begin{aligned}h(Z+\Phi(Z)\,\Delta t+\mathscr{D}Z') ={}& h(Z)+(\mathscr{D}Z')_i\frac{\partial h}{\partial Z_i}+\Delta t\Phi_i(Z)\frac{\partial h}{\partial Z_i}\\ &+\frac{1}{2}(\mathscr{D}Z')_i(\mathscr{D}Z')_j\frac{\partial^2 h}{\partial Z_i\,\partial Z_j}\\ &+\frac{1}{2}\Delta t(\Phi_i(Z)(\mathscr{D}Z')_j+\Phi_j(Z)(\mathscr{D}Z')_i)\frac{\partial^2 h}{\partial Z_i\,\partial Z_j}\\ &+\frac{1}{2}\Delta t^2\Phi_i(Z)\,\Phi_j(Z)\frac{\partial^2 h}{\partial Z_i\,\partial Z_j}+\mathscr{R}^{(3)}\end{aligned} \tag{A.13}$$

where all the derivatives of h are taken at Z, and where the remainder $\mathscr{R}^{(3)}$ is:

$$\mathscr{R}^{(3)} = \varepsilon_i \varepsilon_j \varepsilon_k \left. \frac{\partial^3 h}{\partial Z_i \, \partial Z_j \, \partial Z_k} \right|_{Z + a\varepsilon} \tag{A.14}$$

with $\varepsilon = \Phi(Z) \, \Delta t + \mathscr{D} Z'$ and $0 \leqslant a \leqslant 1$. Let us study now the contribution of the zeroth, first and second order terms of this expansion to the integral in (A.11). The contribution of the remainder with be studied separately afterwards. It will be proven to be of order Δt^2.

Because of property (A.7) all linear terms in $\mathscr{D} Z'$ vanish from the integral.

Because of property (A.8) the quadratic term in $\mathscr{D} Z'$ will contribute to the amount of

$$\Delta t \, D \int_{\mathbb{R}^6} \sum_{i=4}^{6} \left. \frac{\partial^2 h}{\partial Z_i \, \partial Z_i} \right|_Z \Pi(t_0, Z) \, dZ$$

to the integral in (A.11). Note that the summation has been explicited here because the index i has to range between 4 and 6 instead of 1 and 6. This is due to the fact that the first three diagonal elements of $\mathscr{D}$ are zero.

The quadratic term in Δt will not be relevant in the final steps of the derivation.

By using (A.13) in (A.11) and identifying the result with (A.10), one obtains:

$$\begin{aligned} \int_{\mathbb{R}^6} h(Z) \, \Pi(t_0 + \Delta t, Z) \, dZ = & \int_{\mathbb{R}^6} h(Z) \, \Pi(t_0, Z) \, dZ \\ & + \Delta t \int_{\mathbb{R}^6} \Phi_i(Z) \frac{\partial h}{\partial Z_i} \Pi(t_0, Z) \, dZ \\ & + \Delta t \, D \int_{\mathbb{R}^6} \sum_{i=4}^{6} \frac{\partial^2 h}{\partial Z_i \, \partial Z_i} \Pi(t_0, Z) \, dZ \\ & + \mathcal{O}(\Delta t^2) \end{aligned} \tag{A.15}$$

where we used the fact that h is $\mathscr{C}^{\infty}$ with compact support, and that Φ is bounded on any compact.

By expanding Π in Δt in the l.h.s. of (A.15), dividing the resulting equation by Δt and finally taking the limit $\Delta t \to 0$, one obtains:

$$\int_{\mathbb{R}^6} h(Z) \frac{\partial \Pi}{\partial t} dZ = \int_{\mathbb{R}^6} \Phi_i(Z) \frac{\partial h}{\partial Z_i} \Pi(t_0, Z) \, dZ + D \int_{\mathbb{R}^6} \sum_{i=4}^{6} \frac{\partial^2 h}{\partial Z_i \partial Z_i} \Pi(t_0, Z) \, dZ \tag{A.16}$$

Since the support of h is compact one deduces directly by integrating the r.h.s. of (A.16) by parts that:

$$\int_{\mathbb{R}^6} h(Z) \left\{ \frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial Z_i} (\Phi_i \Pi) - D \sum_{i=4}^{6} \frac{\partial^2}{\partial Z_i \partial Z_i} \Pi \right\} dZ = 0 \tag{A.17}$$

Because this last identity has to be true for any h, one is naturally led to Kramers' equation:

$$\frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial Z_i} (\Phi_i \Pi) - D \sum_{i=4}^{6} \frac{\partial^2}{\partial Z_i \partial Z_i} \Pi = 0 \tag{A.18}$$

Giving to $\Phi_i(Z)$ its value $(\mathbf{p}/(\gamma m), -\alpha\gamma\mathbf{p})$, one ends up with Kramers' equation (6) of Section 3.2:

$$\frac{\partial \Pi}{\partial t} + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{p}}{\gamma m} \Pi \right) + \nabla_{\mathbf{p}} \cdot (-\alpha\gamma\mathbf{p}\Pi) = D \Delta_{\mathbf{p}} \Pi$$

Let us now justify why the remainder does not give any constant or linear contribution in Δt. The remainder takes the form:

$$\begin{aligned} \mathscr{R}^{(3)} = \left. \frac{\partial^3 h}{\partial Z_i \partial Z_j \partial Z_k} \right|_{Z + a\varepsilon} & ((\mathscr{D}Z')_i (\mathscr{D}Z')_j (\mathscr{D}Z')_k \\ & + (\Phi_i(Z)(\mathscr{D}Z')_j (\mathscr{D}Z')_k + \text{circ.}) \Delta t \\ & + (\Phi_i(Z) \Phi_j(Z)(\mathscr{D}Z')_k + \text{circ.}) \Delta t^2 \\ & + \Phi_i(Z) \Phi_j(Z) \Phi_k(Z) \Delta t^3) \end{aligned} \tag{A.19}$$

where "circ." designates the circular permutations on the indices i, j and k. The only possible constant contribution vanishes because G is even with respect to its variable Z'. The linear contribution in Δt could only come from the second line in (A.19). Let us for example evaluate the contribution of one of the three terms in this line, namely:

$$M \equiv \Delta t \int_{\mathbb{R}^6} \int_{\mathbb{R}^6} \Phi_i(Z)(\mathscr{D}Z')_j\,(\mathscr{D}Z')_k \left.\frac{\partial^3 h}{\partial Z_i\,\partial Z_j\,\partial Z_k}\right|_{Z+a\varepsilon}$$
$$\times\, \Pi(t_0, Z)\; G(\Delta t, Z')\; dZ\, dZ'$$

This term vanishes because of property (A.7) as soon as $j \neq k$. Therefore, one can write:

$$M = \Delta t \int_{\mathbb{R}^6} dZ\; \Phi_i(Z) \left.\frac{\partial^3 h}{\partial Z_i\,\partial Z_j\,\partial Z_k}\right|_{Z+a\varepsilon} \Pi(t_0, Z)\; \delta_{jk}$$
$$\times \int_{\mathbb{R}^3} d^3x' \int_{\mathbb{R}^3} d^3p'\, 2D\,\frac{p'^2}{3}\, G(\Delta t, \mathbf{x}', \mathbf{p}')$$

The integration over $\mathbf{x}'$ is trivial. By using the new variable $\mathbf{u}' = \mathbf{p}'/\sqrt{\Delta t}$ in the last integral, it is easy to verify that the end result scales as Δt^2.

APPENDIX B. NORMALIZATION AND SECOND ORDER MOMENTUM OF THE RELATIVISTIC MAXWELL–BOLTZMANN DISTRIBUTION

The goals of the present Appendix are (i) to justify the normalization factor of the relativistic Maxwell–Boltzmann distribution (12), (ii) to compute the second order momentum $\langle \mathbf{p}^2 \rangle$ of this distribution, and (iii) to show that $\langle \mathbf{p}^2 \rangle$ indeed increases with the temperature, as in the Galilean case.

Before going any further, we recall the definition and a useful integral expression for the modified Hankel function $K_\nu(z)$ (see ref. 1).

We call "modified Hankel function" and we denote $K_\nu(z)$ the function:

$$K_\nu(z) = \frac{i\pi}{2}\, e^{\nu i\pi/2} H_\nu^{(1)}(iz) \tag{B.20}$$

where $H_\nu^{(1)}(z)$ is a Bessel function of the third kind (Hankel function). The following integral representation of $K_\nu(z)$ will be used in the present Appendix:

$$K_\nu(z) = \int_0^\infty e^{-z\,\mathrm{ch}\,\theta}\, \mathrm{ch}\,\nu\theta\; d\theta$$
$$|\arg(z)| < \frac{\pi}{2}, \qquad \text{or} \quad \mathrm{Re}(z) = 0 \qquad \text{for} \quad \nu = 0 \tag{B.21}$$

In the sequel, the argument of K_ν will be real and strictly positive, and the order ν will be an integer.

Normalization Factor

The relativistic Maxwell–Boltzman (or Jüttner) distribution is proportional to the exponential of the opposite of the relativistic energy divided by kT, where k is the Boltzmann constant and T the absolute temperature. It can thus be expressed in terms of the Lorentz factor γ as:

$$\Pi(\mathbf{p}) = \mathscr{A} e^{-mc^2\gamma/kT} \tag{B.22}$$

or in terms of $\mathbf{p}$ as:

$$\Pi(\mathbf{p}) = \mathscr{A} e^{-\sqrt{\mathbf{p}^2c^2 + m^2c^4}/kT} \tag{B.23}$$

The normalization condition:

$$\int_{\mathbb{R}^3} \Pi(\mathbf{p})\, d^3p = 1 \tag{B.24}$$

where the integral is extended over $\mathbb{R}^3$, implies that:

$$\mathscr{A}^{-1} = \int_0^\infty e^{-\sqrt{p^2c^2 + m^2c^4}/kT} 4\pi p^2\, dp \tag{B.25}$$

We introduce the new variable θ such that $p = mc \operatorname{sh} \theta$ to re-express (B.25) under the following form:

$$\begin{aligned} \mathscr{A}^{-1} &= 4\pi(mc)^3 \int_0^\infty e^{-\beta \operatorname{ch} \theta} \operatorname{sh}^2 \theta \operatorname{ch} \theta\, d\theta \\ &= 4\pi(mc)^3 \int_0^\infty \operatorname{sh} \theta e^{-\beta \operatorname{ch} \theta} \tfrac{1}{2} \operatorname{sh} 2\theta\, d\theta \end{aligned}$$

where β designates the temperature coefficient mc^2/kT. We integrate by parts the above expression to obtain:

$$\mathscr{A}^{-1} = \frac{4\pi(mc)^3}{\beta} \int_0^\infty e^{-\beta \operatorname{ch} \theta} \operatorname{ch} 2\theta\, d\theta$$

The second order modified Hankel function $K_2(\beta)$ comes out naturally and we get finally the normalization factor $\mathscr{A}$ of the relativistic Maxwell–Boltzmann distribution (12):

$$\mathscr{A} = \frac{1}{4\pi(mc)^3} \frac{\beta}{K_2(\beta)} \tag{B.26}$$

Computation of $\langle \mathbf{p}^2 \rangle$

Using (12), we can compute the mean square momentum as follows:

$$\langle \mathbf{p}^2 \rangle = \frac{1}{4\pi(mc)^3} \frac{mc^2/kT}{K_2(mc^2/kT)} \int_0^\infty e^{-\sqrt{p^2c^2+m^2c^4}/kT} p^2 4\pi p^2 \, dp \tag{B.27}$$

As in the previous calculation, we introduce the variable θ such that $p = mc \operatorname{sh} \theta$ and the temperature coefficient $\beta = mc^2/kT$. We then obtain:

$$\begin{aligned}\langle \mathbf{p}^2 \rangle &= (mc)^2 \frac{\beta}{K_2(\beta)} \int_0^\infty e^{-\beta \operatorname{ch} \theta} \operatorname{sh}^4 \theta \operatorname{ch} \theta \, d\theta \\ &= (mc)^2 \frac{\beta}{K_2(\beta)} \int_0^\infty \operatorname{sh} \theta e^{-\beta \operatorname{ch} \theta} \operatorname{sh}^2 \theta \, \tfrac{1}{2} \operatorname{sh} 2\theta \, d\theta\end{aligned}$$

We integrate by parts the above expression to obtain:

$$\begin{aligned}\langle \mathbf{p}^2 \rangle &= (mc)^2 \frac{1}{K_2(\beta)} \int_0^\infty e^{-\beta \operatorname{ch} \theta} (\operatorname{sh} \theta \operatorname{ch} \theta \operatorname{sh} 2\theta + \operatorname{sh}^2 \theta \operatorname{ch} 2\theta) \, d\theta \\ &= (mc)^2 \frac{1}{K_2(\beta)} \int_0^\infty e^{-\beta \operatorname{ch} \theta} \left(\frac{1}{2} \operatorname{sh}^2 2\theta + \frac{1}{2} (\operatorname{ch} 2\theta - 1) \operatorname{ch} 2\theta \right) d\theta \\ &= \frac{(mc)^2}{2} \frac{1}{K_2(\beta)} \int_0^\infty e^{-\beta \operatorname{ch} \theta} (\operatorname{sh}^2 2\theta + \operatorname{ch}^2 2\theta - \operatorname{ch} 2\theta) \, d\theta \\ &= \frac{(mc)^2}{2} \frac{1}{K_2(\beta)} \int_0^\infty e^{-\beta \operatorname{ch} \theta} (\operatorname{ch} 4\theta - \operatorname{ch} 2\theta) \, d\theta\end{aligned}$$

We recognize above the integral expressions of the second and fourth order modified Hankel functions. We then finally get for $\langle \mathbf{p}^2 \rangle$:

$$\langle \mathbf{p}^2 \rangle = \frac{(mc)^2}{2} \left(\frac{K_4(mc^2/kT)}{K_2(mc^2/kT)} - 1 \right) \tag{B.28}$$

Proof That $\langle \mathbf{p}^2 \rangle$ Increases with T

We start with expression (B.28) for $\langle \mathbf{p}^2 \rangle$, and use as before the parameter $\beta = mc^2/kT$. We compute the derivative of $\langle \mathbf{p}^2 \rangle$ with respect to β:

$$\frac{d\langle \mathbf{p}^2 \rangle}{d\beta} = \frac{1}{2}\left(\frac{mc}{K_2(\beta)}\right)^2 \left(K_2 \frac{dK_4}{d\beta} - K_4 \frac{dK_2}{d\beta}\right) \tag{B.29}$$

where the argument β has been omitted in the expression between parentheses. From the integral expression (B.21) of the modified Hankel functions, we obtain their derivatives:

$$\frac{dK_2}{d\beta} = -\int_0^\infty e^{-\beta \operatorname{ch} \theta} \operatorname{ch} 2\theta \operatorname{ch} \theta \, d\theta$$

and

$$\frac{dK_4}{d\beta} = -\int_0^\infty e^{-\beta \operatorname{ch} \theta} \operatorname{ch} 4\theta \operatorname{ch} \theta \, d\theta$$

The expression between parentheses in (B.29) involves products of integrals. It can be written as a double integral:

$$K_2 \frac{dK_4}{d\beta} - K_4 \frac{dK_2}{d\beta} = \int_0^\infty \int_0^\infty e^{-\beta(\operatorname{ch}\theta_1 + \operatorname{ch}\theta_2)} (\operatorname{ch}\theta_1 - \operatorname{ch}\theta_2) \operatorname{ch} 2\theta_1 \operatorname{ch} 4\theta_2 \, d\theta_1 \, d\theta_2$$

$$= \frac{1}{2} \int_0^\infty \int_0^\infty e^{-\beta(\operatorname{ch}\theta_1 + \operatorname{ch}\theta_2)} (\operatorname{ch}\theta_1 - \operatorname{ch}\theta_2)$$
$$\times (\operatorname{ch} 2\theta_1 \operatorname{ch} 4\theta_2 - \operatorname{ch} 2\theta_2 \operatorname{ch} 4\theta_1) \, d\theta_1 \, d\theta_2$$

$$= \frac{1}{2} \int_0^\infty \int_0^\infty e^{-\beta(\operatorname{ch}\theta_1 + \operatorname{ch}\theta_2)} (\operatorname{ch}\theta_1 - \operatorname{ch}\theta_2)$$
$$\times (\operatorname{ch} 2\theta_1 (2 \operatorname{ch}^2 2\theta_2 - 1) - \operatorname{ch} 2\theta_2 (2 \operatorname{ch}^2 2\theta_1 - 1)) \, d\theta_1 \, d\theta_2$$

$$= \frac{1}{2} \int_0^\infty \int_0^\infty e^{-\beta(\operatorname{ch}\theta_1 + \operatorname{ch}\theta_2)} (\operatorname{ch}\theta_1 - \operatorname{ch}\theta_2)$$
$$\times (\operatorname{ch} 2\theta_2 - \operatorname{ch} 2\theta_1)(1 + 2 \operatorname{ch} 2\theta_1 \operatorname{ch} 2\theta_2) \, d\theta_1 \, d\theta_2$$

Since the hyperbolic cosine monotonically increases for positive values of its argument, it becomes clear that the above expression is negative. The mean square momentum $\langle \mathbf{p}^2 \rangle$ thus decreases with β and increases with the temperature T, just as in the Galilean case.

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REFERENCES

1. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products* (Academic Press, 1965), p. 952, formula 8.407.1, and p. 958, formula 8.432.1.
2. M. W. Zemansky and R. H. Dittman, *Heat and Thermodynamics* (McGraw-Hill, 1981).
3. L. Landau and E. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon Press, 1987).
4. C. Truesdell, *Rational Thermodynamics*, 2nd ed. (Springer Verlag, 1984).
5. I. Mueller and T. Ruggeri, *Extended Thermodynamics* (Springer Verlag, 1993).
6. W. Israel, in *Relativistic Fluid Dynamics*, edited by A. Anile and Y. Choquet-Bruhat (Springer-Verlag, 1989).
7. L. Boltzmann, *Wien Ber.* **66**:275 (1872).
8. A. Einstein, *Ann. Phys.* **17**:549 (1905).
9. A. Einstein, *Investigations on the theory of the Brownian movement*, Ph.D. thesis (Dover, 1956).
10. G. E. Uhlenbeck and L. S. Ornstein, *Phys. Rev.* **36**(3) (1930).
11. N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, 1992).
12. S. K. Ma, *Statistical Mechanics* (World Scientific 1985).
13. C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, 1980).
14. R. Wald, *General Relativity* (University of Chicago Press, 1984).
15. U. M. Titulaer, *Physica A* **91**:321 (1978).
16. H. B. Callen and T. A. Welton, *Phys. Rev.* **83**(1):34 (1951).
17. C. W. Oseen, *Ark. f. Mat. Astr. og Fys.* **6**(29) (1910).
18. S. Kaplun and P. A. Lagerstrom, *J. Math. Mech.* **6**:585 (1957).
19. I. Proudman and J. R. A. Pearson, *J. Fluid Mech.* **2**:237 (1957).
20. M. C. Mackey, *Time's Arrow: the Origins of Thermodynamic Behavior* (Springer Verlag, 1992).
21. F. Jüttner, *Ann. Phys.* **34**:856 (1911).
22. W. H. Press, B. P. Flannery, S. A. Teukolsky, and W.T. Vetterlin, *Numerical Recipes* (Cambridge University Press, 1989).
23. J. L. Anderson and E. Nowotny, *Pysica A* **163**:501 (1990).